Congruences of Tangents to a Surface and Derived Congruences.

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Given any family of curves upon a surface S; the tangents to these curves form a rectilinear congruence for which S is one of the focal sheets. The other focal sheet is a determinate surface S_1, and upon it there is a family of curves to which the lines of the congruence are tangent. If tangents are drawn to the curves on S_1, which are the conjugates of the above family, a second congruence is formed with S_1 for one of the focal sheets, and a third surface S_2 for the other. If none of these successive surfaces reduces to a curve, we get in this manner an endless sequence of congruences and at the same time an infinite suite of surfaces with a known conjugate system. In like manner the tangents to the curves on S whose directions are conjugate to the given curves form a congruence and give rise to a surface S_{-1}, so that the sequence extends in both directions. These are the *derived congruences of Darboux.**

In §1 we consider a surface S referred to a conjugate system of lines and show the relations which hold between the rectangular coordinates of a point on this surface and those of the corresponding points on the surfaces S_1 and S_{-1}. From these and similar ones for S_1 and S_{-1} can be derived those for the consecutive surfaces, and so forth. But we are not so much concerned with the general discussion of derived congruences as with the determination of those sequences of which all the congruences are particular congruences of the same kind.

Having found, in the second section, the conditions which must be satisfied in order that the conjugate system on S and S_1 be orthogonal and also on S and S_{-1}, we find that in no case can there be an infinite sequence composed entirely of *congruences of Guichard*, to use the nomenclature of Bianchi. From

* Leçons, Vol. II, p. 16 et seq.

this it follows almost immediately that there cannot exist an infinite suite of normal congruences.

In §3 are determined the conditions which the coefficients of the first fundamental form of S, referred to a conjugate system, must satisfy in order that the tangents to the curves in one family form a congruence of Ribaucour. When a similar determination is made with regard to S_1, it is found that the conditions for this surface reduce to those for S, so that whenever the tangents to the curves in both systems on S form congruences of Ribaucour, all the congruences of the suite are of this kind. These conditions are reduced considerably in form when the conjugate system on S is orthogonal. And the only isothermic surfaces satisfying these conditions are those with the linear elements

$$ds^2 = UV(du^2 + dv^2), \quad ds^2 = e^{UV}(du^2 + dv^2),$$

where U is a function of u alone and V of v alone.

The determination of these surfaces is made in §§4, 5, and in each case it is shown that all the surfaces of this class are developable.

In §6 we consider the case where the tangents to a family of curves on S form a cyclic congruence and find the conditions which the coefficients of the first quadratic form must satisfy, when the surface is referred to these curves and their conjugates. It is found that, when these curves are the lines of curvature, the congruence is also a congruence of Ribaucour. When the congruence is at the same time cyclic and of Ribaucour, there is an infinity of cyclic systems with the lines of the congruence for axes of the circles, and only in this case.

In the last section, cyclic systems of equal circles are considered, and it is shown that there cannot exist an infinite sequence of cyclic congruences for which the corresponding circles are of this kind. Incidentally, we are led to a consideration of congruences for which the developables in one system are cylindrical, and with this the discussion closes.

§1.—*General Formulae.*

Consider a surface S referred to any conjugate system of lines, $u = \text{const.}$, $v = \text{const.}$; then the rectangular coordinates, x, y, z, of its points are particular integrals of an equation of Laplace of the form

$$\frac{\partial^2\theta}{\partial u\,\partial v} + a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} = 0, \tag{1}$$

where a and b are functions of u and v, whose forms are determined when the linear element

$$ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2 \tag{2}$$

is given.

The tangents to the curves $v = \text{const.}$ form a congruence, C, for which S is one of the focal sheets. Let x_1, y_1, z_1 denote the coordinates of the point on the second sheet, S_1, which corresponds to the point (x, y, z) on S. From this definition it follows that

$$x_1 = x + \lambda \frac{\partial x}{\partial u}, \quad y_1 = y + \lambda \frac{\partial y}{\partial u}, \quad z_1 = z + \lambda \frac{\partial z}{\partial u}, \tag{3}$$

where λ is a function of u and v which is determinate and whose form will be found in a moment. Since S_1 is the second focal sheet of C, the lines of the congruence are tangent to the curves $u = \text{const.}$ on S_1, and, consequently, there exists a function μ such that

$$\frac{\partial x_1}{\partial v} = \mu \frac{\partial x}{\partial u}, \quad \frac{\partial y_1}{\partial v} = \mu \frac{\partial y}{\partial u}, \quad \frac{\partial z_1}{\partial v} = \mu \frac{\partial z}{\partial u}, \tag{4}$$

Differentiate the above expression (3) of x_1 with respect to v and equate it to the above (4); this gives, in consequence of (1),

$$\left(\mu - \frac{\partial \lambda}{\partial v} + a\lambda\right) \frac{\partial x}{\partial u} + (\lambda b - 1) \frac{\partial x}{\partial v} = 0.$$

Since this equation must be satisfied by y and z also, it follows that λ and μ must satisfy the equations

$$\mu - \frac{\partial \lambda}{\partial v} + a\lambda = 0, \quad \lambda b - 1 = 0. \tag{5}$$

Then (3) and (4) become

$$\left.\begin{aligned} x_1 &= x + \frac{1}{b} \frac{\partial x}{\partial u}, \\ \frac{\partial x_1}{\partial v} &= -\frac{1}{b^2}\left(\frac{\partial b}{\partial v} + ab\right)\frac{\partial x}{\partial u}, \end{aligned}\right\} \tag{6}$$

and similarly for y_1 and z_1. When S is given, b can be found directly, and, consequently, S_1 can be determined at once and uniquely.

In a similar manner, the tangents to the curves $u = \text{const.}$ on S form a congruence C_{-1} with S for one of the focal sheets and a new surface S_{-1} for the other focal sheet. By analogy, we have that the rectangular coordinates x_{-1}, y_{-1}, z_{-1} are given by the equations

$$\left.\begin{aligned} x_{-1} &= x + \frac{1}{a}\frac{\partial x}{\partial v}, \\ \frac{\partial x_{-1}}{\partial u} &= -\frac{1}{a^2}\left(\frac{\partial a}{\partial u} + ab\right)\frac{\partial x}{\partial v}, \end{aligned}\right\} \tag{7}$$

and similarly for y_{-1} and z_{-1}.

The general expression for (6) is

$$\theta_1 = \theta + \frac{1}{b}\frac{\partial \theta}{\partial u}, \quad \frac{\partial \theta_1}{\partial v} = -\frac{k}{b^2}\frac{\partial \theta}{\partial u}, \tag{8}$$

where k denotes the second invariant of the equation (1). Combining these two equations, we have

$$\theta k = \theta_1 k + b\frac{\partial \theta_1}{\partial v}.$$

From this it follows that, when k is zero, θ_1 is a function of u alone; similarly, when the first invariant h vanishes. Hence the theorem :*

When k vanishes, S_1 is a curve; and when h vanishes, S_{-1} is a curve.

Equation (1) can be written in the form

$$\frac{\partial}{\partial v}\left(\frac{\partial \theta}{\partial u} + b\theta\right) + a\left(\frac{\partial \theta}{\partial u} + b\theta\right) = k\theta,$$

and by (8),

$$\frac{\partial}{\partial v}(b\theta_1) + ab\theta_1 = k\theta.$$

Eliminating θ between this equation and (8), we find the following equation of which x_1, y_1, z_1 are particular solutions,

$$\frac{\partial^2 \theta_1}{\partial u\,\partial v} + a_1\frac{\partial \theta_1}{\partial u} + b_1\frac{\partial \theta_1}{\partial v} = 0,$$

where

$$a_1 = k/b, \quad b_1 = b - \frac{\partial}{\partial u}\log k/b. \tag{9}$$

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* Darboux, Leçons, Vol. II, p. 21.

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Similarly, the coordinates x_{-1}, y_{-1}, z_{-1} of S_{-1} satisfy the equation

$$\frac{\partial^2\theta}{\partial u\,\partial v}+a_{-1}\frac{\partial\theta}{\partial u}+b_{-1}\frac{\partial\theta}{\partial v}=0,$$

where
$$a_{-1}=\frac{h}{a},\quad b_{-1}=a-\frac{\partial}{\partial v}\log h/a. \tag{10}$$

Proceeding in this manner step by step, the equations corresponding to all the surfaces of the suite of congruences can be found and the coefficients expressed in terms of a, b and their derivatives. Without developing any further the general subject of derived congruences, we pass to a consideration of the more important particular kinds of congruences and the congruences derived from them by the preceding methods.

§ 2.—*Congruences of Guichard. Normal Congruences.*

For the congruence C to be a congruence of Guichard, it is necessary and sufficient that the conjugate systems $u=$ const., $v=$ const. on S and S_1 be orthogonal. Let S be referred to its lines of curvature and denote by α_1, β_1, γ_1 and α_2, β_2, γ_2, the direction cosines of the tangents to the curves $v=$ const., $u=$ const. respectively, and by X, Y, Z, the direction cosines of the normal to S. Between these functions the following relations hold:*

$$\frac{\partial\alpha_1}{\partial u}=-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\alpha_2+\frac{D}{\sqrt{E}}X,\quad \frac{\partial\alpha_1}{\partial v}=\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\alpha_2,$$
$$\frac{\partial\alpha_2}{\partial u}=\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\alpha_1\quad,\quad \frac{\partial\alpha_2}{\partial v}=-\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\alpha_1+\frac{D''}{\sqrt{G}}X,$$

and similarly for β_1, γ_1, β_2, γ_2, where, as usual,

$$D=\Sigma X\frac{\partial^2 x}{\partial u^2},\quad D''=\Sigma X\frac{\partial^2 x}{\partial v^2}.$$

Denoting by 2ρ the focal distance for the congruence, we have

$$x_1=x+2\rho\alpha_1,\quad y_1=y+2\rho\beta_1,\quad z_1=z+2\rho\gamma_1.$$

Differentiating these expressions with respect to u and v respectively, and

* Bianchi, Lezioni, p. 94.

making use of the above relations, we get

$$\frac{\partial x_1}{\partial u} = \left(\sqrt{E} + 2\frac{\partial \rho}{\partial u}\right)\alpha_1 + 2\rho\left(-\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}\alpha_2 + \frac{D}{\sqrt{E}}X\right),$$

$$\frac{\partial x_1}{\partial v} = 2\frac{\partial \rho}{\partial v}\alpha_1 + \alpha_2\left(\sqrt{G} + \frac{2\rho}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}\right),$$

and similarly for y_1 and z_1. Comparing the second of these equations with (4), we see that we must have

$$\sqrt{G} + \frac{2\rho}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u} = 0. \tag{11}$$

Again, if the parametric lines on S_1 are to be orthogonal, it is necessary that

$$2\frac{\partial \rho}{\partial u} + \sqrt{E} = 0. \tag{12}$$

Differentiating (11) with respect to u, we find, in consequence of (12),

$$\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}\right] = 0. \tag{13}$$

In a similar manner we find that the necessary and sufficient condition that the tangents to the lines of curvature $u = \text{const.}$ shall form a congruence of Guichard is

$$\frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}\right] = 0. \tag{14}$$

When a surface is referred to its lines of curvature, the Gauss equation reduces to*

$$K = -\frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}\right]\right\}, \tag{15}$$

where K denotes the total curvature. From this it follows that for the conditions (13) and (14) to be satisfied simultaneously, it is necessary that S be a developable surface. But when S is a developable surface, one family of lines of curvature is composed of the rectilinear generatrices and hence the corresponding congruence does not exist. Hence the theorem:

In no case do the tangents to the lines of curvature in each system on a surface form congruences of Guichard.

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* Bianchi, l. c., p. 67.

And, as a consequence, the theorem:

There does not exist a series of derived congruences all of which are congruences of Guichard.

Bianchi has shown* that when S is one of the focal sheets of a congruence of Guichard, or, as he calls it, a surface of Guichard, one of the sheets of its evolute is a Voss surface, and, moreover, that the conjugate geodesics on the latter correspond to the lines of curvature upon the former. In consequence of the preceding theorems, we have that in no case are both the sheets of the evolute of a surface of Guichard surfaces of Voss.

Let S be a surface of Voss and let the parametric curves be the conjugate system of geodesics. The tangents to the latter form a normal congruence and, as Bianchi has shown,† all the surfaces cutting these lines orthogonally, are surfaces of Guichard. Since the congruence is normal, the curves $u =$ const. on S_1 are geodesics, but the curves $v =$ const. are not geodesics in consequence of the above results. Thus C and C_{-1} are normal congruences, and C_1 is not normal. Hence the theorem:

There cannot exist a derived suite of normal congruences; and, for two consecutive congruences to be normal, the common focal sheet must be a surface of Voss.

§3.—*Congruences of Ribaucour.*

From (6) we have that, if S is the first focal sheet of a congruence, the second sheet S_1 is given by

$$x_1 = x + \frac{1}{b}\frac{\partial x}{\partial u}, \quad y_1 = y + \frac{1}{b}\frac{\partial y}{\partial u}, \quad z_1 = z + \frac{1}{b}\frac{\partial z}{\partial u}. \tag{16}$$

From this it follows that the coordinates, $\bar{x}$, $\bar{y}$, $\bar{z}$, of the mean point of the line have the expressions

$$\bar{x} = x + \frac{1}{2b}\frac{\partial x}{\partial u}, \quad \bar{y} = y + \frac{1}{2b}\frac{\partial y}{\partial u}, \quad \bar{z} = z + \frac{1}{2b}\frac{\partial z}{\partial u}. \tag{17}$$

From this we get by differentiation with respect to u and v, and ready reductions by means of these equations themselves and (1), the following:

$$\frac{\partial^2 \bar{x}}{\partial u \partial v} + \left(a + \frac{1}{b}\frac{\partial b}{\partial v}\right)\frac{\partial \bar{x}}{\partial u} + b\frac{\partial \bar{x}}{\partial v} = -\frac{1}{2b}\left(\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u \partial v}\right)\frac{\partial x}{\partial u}, \tag{18}$$

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* Loc. cit., p. 271. † Ib.

and similarly in $\bar{y}$ and $\bar{z}$. The *congruences of Ribaucour* may be defined as those for which the developables meet the mean surface of the congruence in a conjugate system. From the above discussion it is clear that the ruled surfaces $u = \text{const}$, $v = \text{const.}$ are the developables, and, consequently, it follows from (18) that the necessary and sufficient condition that the tangents to the curves $v = \text{const.}$ on S form a congruence of Ribaucour is that the functions a and b satisfy the condition

$$\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u \partial v} = 0 . \tag{19}$$

In a similar manner we find that for the tangents to the curves $u = \text{const.}$ to form a congruence of this kind, it is necessary and sufficient that

$$\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} - \frac{\partial^2 \log a}{\partial u \partial v} = 0 . \tag{20}$$

Subtracting these two equations of condition we get

$$\frac{\partial^2 \log ab}{\partial u \partial v} = 0 ,$$

so that for the tangents to the curves of both families to be congruences of Ribaucour it is necessary that

$$ab = UV ,$$

where U is a function of u alone and V of v alone. However, this is not the sufficient condition. Solving for a and substituting in (19), we have

$$\frac{\partial^2 \log b}{\partial u \partial v} - \frac{\partial b}{\partial v} - \frac{UV}{b^2}\frac{\partial b}{\partial u} + \frac{U'V}{b} = 0 , \tag{21}$$

where the prime denotes differentiation.

It is well known that any three independent solutions of an equation of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} + a\frac{\partial \theta}{\partial u} + b\frac{\partial \theta}{\partial v} = 0 ,$$

give the rectangular coordinates of a surface S upon which the curves $u = \text{const.}$, $v = \text{const.}$ form a conjugate system. If now b is chosen arbitrarily and a is determined by quadrature from (19), every surface given by solutions of the above equation will be the focal surface of a congruence of Ribaucour. And, if

b is chosen so as to satisfy (21), each surface will be a focal sheet of two congruences of this kind.

When the condition (19) is satisfied, the point equation of the mean surface of C becomes

$$\frac{\partial^2\theta}{\partial u\partial v} + \left(a + \frac{\partial \log b}{\partial v}\right)\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} = 0 .$$

Moreover, this equation has equal invariants in consequence of (19) so that we have the theorem:

The conjugate system in which a congruence of Ribaucour cuts the mean surface has equal invariants.

The condition that the point equation of S may have equal invariants is

$$\frac{\partial a}{\partial u} = \frac{\partial b}{\partial v} .$$

Hence, if the tangents to the curves $v = \text{const.}$ are to form a congruence of Ribaucour, we must have in consequence of (19),

$$b = UV,$$

where U and V are arbitrary functions of u and v respectively. Now

$$\frac{\partial a}{\partial u} = \frac{\partial b}{\partial v} = UV',$$

so that

$$a = V'\int U du + V_1 .$$

If, in particular, V_1 is zero, condition (20) also is satisfied. Hence, for all surfaces whose rectangular coordinates satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial u\,\partial v} + UV'\frac{\partial\theta}{\partial u} + U'V\frac{\partial\theta}{\partial u} = 0,$$

the tangents to the curves $v = \text{const.}$ and to the curves $u = \text{const.}$ form congruences of Ribaucour. When, in particular, U and V are constant, the surfaces S of this class are the so-called *surfaces of translation.*

The necessary and sufficient condition that the tangents to the curves $u = \text{const.}$ on S_1 form a congruence of Ribaucour is

$$\frac{\partial a_1}{\partial u} - \frac{\partial b_1}{\partial v} - \frac{\partial^2 \log a_1}{\partial u\,\partial v} = 0 .$$

When a_1 and b_1 are replaced by their expressions from (9), this condition reduces to (19), as was to have been expected. Again, the condition that the tangents to the curves $v = \text{const.}$ on S_1 form a congruence of Ribaucour is

$$\frac{\partial a_1}{\partial u} - \frac{\partial b_1}{\partial v} + \frac{\partial^2 \log b_1}{\partial u \partial v} = 0,$$

which can be reduced by means of (9) to

$$\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u \partial v} + \frac{\partial^2 \log}{\partial u \partial v}\left(\frac{k}{b}\right) + \frac{\partial^2}{\partial u \partial v} \log\left[b - \frac{\partial}{\partial u} \log \frac{k}{b}\right] = 0.$$

If the condition (19) is satisfied, this equation can be reduced to (20). On account of the symmetry in all this work, we have the theorem:

Whenever the tangents to a system of lines upon a surface form a congruence of Ribaucour, and also the tangents to the conjugate curves form such a congruence, all of the derived congruences are congruences of Ribaucour.

The expressions for a and b in terms of the coefficients of the linear element of S are*

$$a = \frac{F\dfrac{\partial G}{\partial u} - G\dfrac{\partial E}{\partial v}}{2(EG - F^2)}, \qquad b = \frac{F\dfrac{\partial E}{\partial v} - E\dfrac{\partial G}{\partial u}}{2(EG - F^2)}, \tag{22}$$

so that when the conjugate system on S is composed of the lines of curvature, a and b have the expressions

$$a = -\frac{\partial \log \sqrt{E}}{\partial v}, \quad b = -\frac{\partial \log \sqrt{G}}{\partial u}. \tag{23}$$

From these forms and (19) it follows that the necessary and sufficient condition that the tangents to the lines of curvature $v = \text{const.}$ on S form a congruence of Ribaucour, when S is an isothermic surface, is

$$\frac{\partial^2 \log b}{\partial u \partial v} = 0,$$

whence it follows that the linear element can be reduced, by a suitable choice of parameters, to the form

$$ds^2 = e^{UV} V_1 (du^2 + dv^2), \tag{24}$$

* Bianchi, l. c., p. 88.

where U is a function of u alone, and V and V_1 of v alone. In a similar manner it can be shown that the necessary and sufficient condition that S be an isothermic surface with the tangents to the lines of curvature $u =$ const. forming a congruence of Ribaucour, is that the linear element be reducible to the form

$$ds^2 = e^{UV} U_1 (du^2 + dv^2) , \tag{25}$$

when the lines of curvature are parametric. Combining the above results, we have the theorem:

The necessary and sufficient condition that a surface be isothermic and the congruences formed by the tangents to the lines of curvature in each system be congruences of Ribaucour is that the linear element be reducible to either of the forms

$$ds^2 = UV (du^2 + dv^2) , \tag{26}$$

$$ds^2 = e^{UV} (du^2 + dv^2) . \tag{27}$$

When the linear element takes the first form, it follows from (23) that

$$a = -\tfrac{1}{2} \frac{V'}{V} , \quad b = -\tfrac{1}{2} \frac{U'}{U} .$$

From (5) we remark that the focal distance for the congruence C is infinite, if V is constant; and if U is a constant, the surface S_{-1} is at infinity. Similar results follow for the case where the linear element has the form (27).

Consider for a moment the case where V is constant; then both the above linear elements reduce to the form

$$ds^2 = U (du^2 + dv^2) .$$

Then S is a surface of revolution and $v =$ const. are the meridians. In consequence of the preceding discussion, we have the theorem:

The tangents to the meridians of a surface of revolution form a normal congruence of Ribaucour.

We proceed now to the determination of the surfaces with the linear elements (26) and (27).

§4.—*Surfaces with the linear element* $ds^2 = UV (du^2 + dv^2)$.

Let S be a surface with the linear element (26) and the lines of curvature parametric. In order to find the surfaces of this kind, we make use of the

methods followed by Bonnet in his celebrated memoir *Sur les surfaces applicables.**

Put
$$\frac{\sqrt{E}}{\rho_{gv}} = M, \quad \frac{\sqrt{G}}{\rho_{gu}} = N, \quad \frac{\sqrt{E}}{\rho_1} = P, \quad \frac{\sqrt{G}}{\rho_2} = Q, \tag{28}$$

where ρ_{gv}, ρ_{gu} are the radii of geodesic curvature of the lines $v = \text{const.}$, $u = \text{const.}$ respectively, and ρ_1, ρ_2 are the principal radii of normal curvature corresponding to these respective directions. Bonnet shows that the above functions must satisfy the equations

$$\left.\begin{aligned} \frac{\partial P}{\partial v} + MQ = 0, \quad \frac{\partial Q}{\partial u} - NP = 0, \\ \frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} = PQ, \end{aligned}\right\} \tag{29}$$

and conversely that there exists a unique surface corresponding to each set of functions satisfying these equations. By means of the third the first two can be replaced by

$$\begin{aligned} \frac{\partial P^2}{\partial v} &= -2PMQ = -2M\frac{\partial M}{\partial v} + 2M\frac{\partial N}{\partial u}, \\ \frac{\partial Q^2}{\partial u} &= \quad 2NPQ = -2N\frac{\partial N}{\partial u} + 2N\frac{\partial M}{\partial v}, \end{aligned}$$

or

$$\left.\begin{aligned} \frac{\partial}{\partial v}(P^2 + M^2) = 2M\frac{\partial N}{\partial u}, \\ \frac{\partial}{\partial u}(Q^2 + N^2) = 2N\frac{\partial M}{\partial v}. \end{aligned}\right\} \tag{30}$$

By a suitable choice of parameters, the linear element (26) can be put in the form

$$ds^2 = uv\left(\frac{du^2}{U} + \frac{dv^2}{V}\right), \tag{31}$$

which is more convenient for our discussion. Now

$$\begin{aligned} M &= -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} = -\frac{1}{2v}\sqrt{\frac{V}{U}}, \\ N &= \quad \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} = \frac{1}{2u}\sqrt{\frac{U}{V}}. \end{aligned}$$

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* **Journal de l'École Polytechnique, XLII Cahier, p. 44 et seq.**

The equations (30) take the form

$$\frac{\partial}{\partial v}(P^2 + M^2) = -\frac{1}{2uvU}\left(\frac{U'}{2} - \frac{U}{u}\right),$$
$$\frac{\partial}{\partial u}(Q^2 + N^2) = -\frac{1}{2uvV}\left(\frac{V'}{2} - \frac{V}{v}\right),$$

from which it follows that

$$\left.\begin{aligned} P^2 &= \tfrac{1}{2}\frac{1}{uU}\left(\frac{U}{u} - \frac{U'}{2}\right)\log v - \frac{V}{4v^2U} + \frac{U_2}{4U}, \\ Q^2 &= \tfrac{1}{2}\frac{1}{vV}\left(\frac{V}{v} - \frac{V'}{2}\right)\log u - \frac{U}{4u^2V} + \frac{V_2}{4V}, \end{aligned}\right\} \tag{32}$$

where U_2 and V_2 are functions of u and v respectively, whose forms must be determined. If we square the last of equations (29) and substitute the above expressions for M, N, P, Q, this equation becomes

$$\left(\frac{V'}{v} - \frac{2V}{v^2} + \frac{U'}{u} - \frac{2U}{u^2}\right)^2 = \left[\left(\frac{2U}{u^2} - \frac{U'}{u}\right)\log v - \frac{V}{v^2} + U_2\right]\left[\left(\frac{2V}{v^2} - \frac{V'}{v}\right)\log u - \frac{U}{u^2} + V_2\right].$$

If we put

$$\frac{V}{v^2} = V_1, \quad \frac{U}{u^2} = U_1, \tag{33}$$

the above equation can be written in the form

$$(vV_1' + uU_1')^2 = (uU_1'\log v + V_1 - U_2)(vV_1'\log u + U_1 - V_2). \tag{34}$$

We consider first the case where one of the factors on the right, say the former, is equal to zero. From its form we see that we must have

$$uU_1' = a, \quad U_2 = b, \quad V_1 = b - a\log v,$$

where a and b are arbitrary constants. Moreover, for these values the left-hand member vanishes. Hence a solution is given by

$$U_1 = \beta + a\log u, \quad V_1 = b - a\log v, \quad U_2 = b, \quad V_2 \text{ arbitrary}. \tag{35}$$

But in this case P is zero, and hence from (28) we see that S is a developable surface. Similar results follow if the second factor vanishes.

Suppose now that neither of the right-hand factors vanish, and for the sake of brevity put

$$\theta = vV_1' + uU_1', \quad A = uU_1' \log v + V_1 - U_2, \quad B = vV_1' \log u + U_1 - V_2; \quad (36)$$

then equation (34) may be written

$$\frac{\theta^2}{A} = B. \quad (37)$$

Differentiating with respect to u and making use of the notation

$$p, q, r, t = \frac{\partial\theta}{\partial u}, \quad \frac{\partial\theta}{\partial v}, \quad \frac{\partial^2\theta}{\partial u^2}, \quad \frac{\partial^2\theta}{\partial v^2},$$

we get
$$\frac{2\theta p}{A} - \frac{\theta^2(p \log v - U_2')}{A^2} = \frac{\theta}{u}.$$

As we have excluded the case where $\theta = 0$, this may be written

$$\frac{A^2}{\theta u} - 2\frac{pA}{\theta} + p \log v = U_2',$$

which, upon differentiation with respect to v, becomes

$$A^2\frac{q}{u} - 2A\left(\frac{\theta^2}{uv} + pq\right) + \frac{p\theta^2}{v} = 0.$$

From this equation we have

$$A\frac{q}{u} = \frac{\theta^2}{uv} + pq \pm \sqrt{\frac{\theta^4}{u^2v^2} + \frac{pq\theta^2}{uv} + p^2q^2}, \quad (38)$$

If $p = 0$, this becomes

$$\frac{Aq}{u} = \frac{\theta^2}{uv} \pm \frac{\theta^2}{uv},$$

whence
$$A = 0, \quad q = 0, \quad \text{or} \quad Aq = \frac{2\theta^2}{v}.$$

The first case has been excluded. For the second to hold we must have

$$uU_1' = a, \quad vV_1' = a,$$

where a and α are arbitrary constants, and equation (34) becomes

$$(\alpha + a)^2 = [(\alpha + a)\log v + b - U_2][(\alpha + a)\log u + \beta - V_2].$$

Put $(\alpha + a)\log v = v_1$, $(\alpha + a)\log u = u_1$, $b - U_2 = U_3$, $\beta - V_2 = V_3$, then the above equation becomes

$$(\alpha + a)^2 = [v_1 + U_3][u_1 + V_3].$$

Differentiating with respect to u_1, we get

$$v_1 + U_3 + U_3'(u_1 + V_3) = 0,$$

which evidently is impossible. Hence the second case cannot arise, unless $a = -\alpha$, and this is the first case. For the third case, equation (37) becomes

$$B = \frac{qv}{2},$$

which, upon differentiation with respect to u, gives $\theta = 0$. Hence p cannot vanish unless θ vanishes.

On account of the symmetry of the expression (38) for $\frac{Aq}{u}$, it follows that $\frac{Bp}{v}$ has the same expression, hence we can put

$$\frac{Aqv}{pu} = B,$$

Taking the derivative with respect to u, we get

$$\frac{(p\log v - U_2')\,qv}{pu} - \frac{Aqv\,(p + ur)}{p^2u^2} = \frac{\theta}{u},$$

which may be written

$$\frac{\log v}{u} - \frac{A\,(p + ur)}{p^2u^2} - \frac{\theta}{uvq} = \frac{U_2'}{pu}.$$

When this is differentiated with respect to v, it becomes

$$\frac{r}{p^2} + \frac{1}{pu} = \frac{t}{q^2} + \frac{1}{qv}.$$

From the expression for θ, it follows that the left-hand member of the above

equation does not involve v, nor the right-hand member u, so that each must be a constant; consequently, this equation can be replaced by the two

$$\frac{dp}{du} + \frac{p}{u} + \alpha p^2 = 0, \quad \frac{dq}{dv} + \frac{q}{v} + \alpha q^2 = 0. \tag{39}$$

where α is constant.

For $\alpha = 0$, these equations give

$$p = \frac{2\beta}{u}, \quad q = \frac{2b}{v},$$

and
$$\theta = 2\beta \log u + \gamma + 2b \log v + c.$$

From the definition of θ we get

$$\left.\begin{aligned} U_1 &= \beta \log^2 u + \gamma \log u + \delta, \\ V_1 &= b \log^2 v + c \log v + d, \end{aligned}\right\} \tag{40}$$

where $\beta, \gamma, \ldots, d$ are constants whose values must be such that equation (34) will be satisfied. If we substitute these values in (34) and, for the sake of brevity, put

$$u_1 = \log u, \quad v_1 = \log v, \quad U_3 = d - U_2, \quad V_3 = \delta - V_2,$$

we get

$$\begin{aligned} &(2\beta u_1 + 2bv_1 + \gamma + c)^2 \\ &\quad = [2bu_1v_1 + (\gamma + c)u_1 + \beta u_1^2 + V_3][2\beta u_1v_1 + (\gamma + c)v_1 + bv_1^2 + U_3], \end{aligned} \tag{41}$$

Differentiating with respect to u_1 and v_1, and again with respect to these two parameters, we get

$$U_3'' V_3'' + 20b\beta = 0,$$

so that we may put

$$U_3'' = 2k\beta, \quad V_3'' = -\frac{10b}{k},$$

where k is a constant different from zero. From these equations we get

$$U_3 = k\beta u^2 + \lambda u + \mu, \quad V_3 = -\frac{5bv^2}{k} + \rho v + \sigma,$$

where $\lambda, \mu, \rho, \sigma$ are constants, such that when these expressions for U_3 and V_3

are substituted in (41), it will vanish identically. When this substitution is made, it is found that the coefficient of u_1^4 is $k\beta^2$ and of v_1^4 is $-\frac{5b^2}{k}$. Hence we must have $\beta = b = 0$. When these values are substituted (41), it is readily found that $\gamma + c = 0$, so that *formulae* (40) *reduce to* (35).

We consider finally the case where $\alpha \neq 0$. From (39) we find that

$$\frac{1}{pu} = \beta + \alpha \log u, \quad \frac{1}{qv} = b + \alpha \log v,$$

so that

$$\theta = \frac{1}{\alpha} \log (\beta + \alpha \log u) + \gamma + \frac{1}{\alpha} \log (b + \alpha \log v) + c,$$

and

$$\left.\begin{aligned} U_1 &= \frac{1}{\alpha^2} [(\beta + \alpha \log u) \log (\beta + \alpha \log u) - \alpha \log u] + \gamma \log u + \delta, \\ V_1 &= \frac{1}{\alpha^2} [(b + \alpha \log v) \log (b + \alpha \log v) - \alpha \log v] + c \log v + d. \end{aligned}\right\} \quad (42)$$

We substitute these values in equation (34) and, as before, put

$$u_1 = \log u, \quad v_1 = \log v, \quad U_3 = d - U_2, \quad V_3 = \delta - V_2;$$

then the equation becomes

$$\begin{aligned} &\left[\frac{1}{\alpha} \log (\beta + \alpha u_1) + \gamma + \frac{1}{\alpha} \log (b + \alpha v_1) + c\right]^2 \\ &\qquad = \left\{\left[c + \frac{1}{\alpha} \log (b + \alpha v_1)\right] u_1 + U_1 + V_3\right\} \\ &\qquad\quad \left\{\left[\gamma + \frac{1}{\alpha} \log (\beta + \alpha u_1)\right] v_1 + V_1 + U_3\right\}. \end{aligned} \quad (42')$$

Differentiate this equation with respect to u_1 and v_1; multiply by $(\beta + \alpha u_1)(b + \alpha v)$; differentiate the result twice with respect to u_1 and once with respect to v_1. This gives the equation

$$\begin{aligned} &[(b + \alpha v_1) V_3'' + \alpha V_3'][(\beta + \alpha u_1) U_3''' + 2\alpha U_3''] \\ &\qquad + \frac{2\alpha^2}{\beta + \alpha u_1}\left[c + \frac{1}{\alpha} \log (b + \alpha v_1) + \gamma + \frac{1}{\alpha} \log (\beta + \alpha u_1)\right] \\ &\qquad + \frac{6\alpha}{\beta + \alpha u_1} = 0. \end{aligned} \quad (43)$$

If this equation be differentiated with respect to v_1, it becomes

$$[(b+av_1)V_3''''+2aV_3''][(\beta+au_1)U_3''''+2aU_3'']+\frac{2a^2}{(\beta+au_1)(b+av_1)}=0.$$

In accordance with this equation, we put

$$\left.\begin{aligned}(\beta+au_1)U_3'''+2aU_3''&=\frac{ak}{\beta+au_1},\\(b+av_1)V_3'''+2aV_3''&=-\frac{2a}{k(b+av_1)},\end{aligned}\right\}\tag{44}$$

where k is a constant different from zero. From the second of these equations we have by integration

$$(b+av_1)V_3''+aV_3'=-\frac{2}{k}\log(b+av_1)+\lambda.$$

When this expression and the first of (44) are substituted in (43), the latter becomes

$$\frac{2a}{\beta+au_1}\left[\log(\beta+au_1)+\frac{k\lambda}{2}+3-a(\gamma+c)\right]=0.$$

From this it follows that $a=0$. Hence *the formulae* (35) *give the only solution of the problem.* We shall consider this case for a moment.

From (33) and (35) we have

$$U=u^2(\beta+a\log u),\quad V=v^2(b-a\log v),$$

and from (32),

$$P=0,\quad Q^2=\frac{V_2-b}{4V},$$

so that Q is an arbitrary function of v.

The linear element takes the form

$$ds^2=uv\left[\frac{du^2}{u^2(\beta+a\log u)}+\frac{dv^2}{v^2(b-a\log v)}\right].$$

By a suitable choice of parameters, this can be changed to

$$ds^2=e^{a(u^2-v^2)}(du^2+dv^2),\tag{45}$$

and from (28) we have

$$\frac{1}{\rho_1}=0,\quad\frac{1}{\rho_2}=e^{-\frac{au^2}{2}}V,$$

where V is an arbitrary function of v. When $a=0$, the linear element becomes

$$ds^2 = du^2 + dv^2,$$

and since ρ_2 is an arbitrary function of v alone, this class comprises all the cylinders.

§5.—*Surfaces with the linear element* $ds^2 = e^{UV}(du^2+dv^2)$.

The discussion of this case is in every way similar to the preceding, so that we shall merely indicate the steps.

The linear element will be taken in the general form

$$ds^2 = e^{uv}\left(\frac{du^2}{U}+\frac{dv^2}{V}\right).$$

The functions M, N, P, Q have the following forms:

$$M = -\frac{v}{2}\sqrt{\frac{V}{U}}, \quad P^2 = -\frac{v^2}{8U}(2U+uU') + \frac{U_2}{4U} - \frac{v^2 V}{4U},$$

$$N = \frac{u}{2}\sqrt{\frac{U}{V}}, \quad Q^2 = -\frac{1}{4V}\left[\frac{u^2}{2}(2V+vV') + u^2 U - V_2\right].$$

Put $$v^2 V = V_1, \quad u^2 U = U_1, \tag{46}$$

then the last of equations (29) becomes

$$\left(\frac{V_1'}{v}+\frac{U_1'}{u}\right)^2 = \left(\frac{v^2}{2}\,\frac{U_1'}{u} + V_1 - U_2\right)\left(\frac{u^2}{2}\,\frac{V_1'}{v} + U_1 - V_2\right). \tag{47}$$

For either of the factors on the right to vanish, that is, for S to be a developable surface of the class considered, we must have

$$U_1 = \alpha u^2 + \beta, \quad V_1 = -\alpha v^2 + \gamma, \quad U_2 = \gamma, \quad V_2 \text{ arbitrary}. \tag{48}$$

Excluding this case, we put

$$\theta = \frac{V_1'}{v} + \frac{U_1}{u}, \quad A = \frac{v^2}{2}\,\frac{U_1'}{u} + V_1 - U_2, \quad B = \frac{u^2}{2}\,\frac{V_1'}{v} + U_1 - V_2,$$

and then write the equation in the form

$$\frac{\theta^2}{A} = B.$$

Differentiating this equation with respect to u and v so as to eliminate U_2 and V_2, we get finally

$$Auq = \theta^2 uv + pq \pm \sqrt{\theta^4 u^2 v^2 + pquv\theta^2 + p^2 q^2}. \tag{49}$$

As before, we can show that p can vanish only when θ is zero.

Again we remark that the right-hand member of (49) must be the expression for Bvp also, so that we have

$$\frac{Auq}{vp} = B,$$

from which we get, by differentiating with respect to u and v,

$$\frac{r}{p^2} - \frac{1}{pu} = \frac{t}{q^2} - \frac{1}{qv}.$$

Since the left-hand member does not involve v and the right-hand u, this equation may be replaced by the two

$$\frac{1}{p^2}\frac{dp}{du} - \frac{1}{pu} - 2a = 0, \quad \frac{1}{q^2}\frac{dq}{dv} - \frac{1}{qv} - 2a = 0.$$

Consider first the case where a is zero. Then we get

$$\theta = 4\beta u^2 + 2\gamma + 4bv^2 + 2c,$$

and
$$U_1 = \beta u^4 + \gamma u^2 + \delta, \quad V_1 = bv^4 + cv^2 + d.$$

If we substitute these expressions in (47) and replace u^2 by u_1 and v^2 by v_1, this equation will become

$$\begin{aligned} 4(2\beta u_1 + \gamma + 2bv_1 + c)^2 = [&2bu_1v_1 + (\gamma + c)u_1 \\ &+ \beta u_1^2 + \delta - V_2][2\beta u_1 v_1 + (\gamma + c)v_1 + bv_1^2 + d - U_2]. \end{aligned}$$

When this is compared with (41), it is seen that the left-hand member differs by the factor 4, and the right-hand member is the same. Hence we are brought to the result that when a is zero in the above equations, no new solutions are given.

Finally, when a is different from zero,

$$\theta = \frac{1}{2a}\log(\beta + au^2) + 2\gamma + \frac{1}{2a}\log(b + av^2) + 2c,$$

and
$$U_1 = \frac{1}{4a^2}\left[(au^2+\beta)\log(au^2+\beta) - au^2\right] + \gamma u^2 + \delta\,,$$
$$V_1 = \frac{1}{4a^2}\left[(av^2+b)\log(av^2+b) - av^2\right] + cv^2 + d\,.$$

When these values are substituted in (47) and $2a$ is replaced by α, u^2 by $2u_1$, v^2 by $2v_1$, 2γ by γ, and $2c$ by c, we get (42′). Hence, as in the preceding case, *the only surfaces with the given linear element are developable, satisfying the equations* (48).

For these developable surfaces we have

$$P = 0\,, \quad Q = \frac{V_2 - \beta}{4V}\,,$$

so that Q is arbitrary, and the linear element can be brought, by a suitable choice of parameters, to the form

$$ds = e^{\sqrt{au^2+\beta}\,.\,\sqrt{b-av^2}}\,(du^2 + dv^2)\,, \tag{50}$$

and the principal radii are given by

$$\frac{1}{\rho_1} = 0\,, \quad \frac{1}{\rho_2} = e^{-\frac{1}{2}\sqrt{au^2+\beta}\,.\,\sqrt{b-av^2}}\,V.$$

As before, we note that when α is zero, the above linear element becomes

$$ds^2 = du^2 + dv^2,$$

and all the surfaces are cylindrical.

We have thus seen that all the surfaces whose linear elements take either of the forms (26) and (27) are developable. But one family of the lines of curvature of a developable surface is composed of the rectilinear generatrices and hence the tangents to these curves form a ruled surface and not a congruence. Gathering together all these results, we have the theorem:

Of all the surfaces for which the tangents to the lines of curvature in both systems form congruences of Ribaucour, none are isothermic.

§6.—*Cyclic Congruences.*

Let S be referred to any conjugate system of lines, and upon the tangents to the curves $v = \text{const.}$ as axes construct circles of radius R and center $\bar{x}$, $\bar{y}$, $\bar{z}$.

Then these coordinates will be given by expressions of the form

$$\bar{x} = x + t\frac{\partial x}{\partial u}, \quad \bar{y} = y + t\frac{\partial y}{\partial u}, \quad \bar{z} = z + t\frac{\partial z}{\partial u},$$

and the coordinates of a point on the circle have the values

$$\xi = \bar{x} + R(\alpha_1 \cos\theta + \alpha_2 \sin\theta), \quad \eta = \bar{y} + R(\beta_1 \cos\theta + \beta_2 \sin\theta),$$
$$\zeta = \bar{z} + R(\gamma_1 \cos\theta + \gamma_2 \sin\theta),$$

where $\alpha_1, \beta_1, \gamma_1$ denote the direction-cosines of the line of intersection of the tangent plane to S at (x, y, z) and the plane of the circle; $\alpha_2, \beta_2, \gamma_2$ are the direction cosines of the line in the latter plane and at right angles with the tangent plane; and θ is the angle of inclination of the radius and the former line. Bianchi* shows that the necessary and sufficient condition that there may exist a family of surfaces which cut these circles orthogonally is that the following relations be satisfied:

$$R^2\left[\frac{\partial}{\partial v}\Sigma\alpha_2\frac{\partial\alpha_1}{\partial u} - \frac{\partial}{\partial u}\Sigma\alpha_2\frac{\partial\alpha_1}{\partial v}\right] + \Sigma\alpha_1\frac{\partial\bar{x}}{\partial u}\,.\,\Sigma\alpha_2\frac{\partial\bar{x}}{\partial v} - \Sigma\alpha_2\frac{\partial\bar{x}}{\partial v}$$
$$- \Sigma\alpha_1\frac{\partial\bar{x}}{\partial v}\,.\,\Sigma\alpha_2\frac{\partial\bar{x}}{\partial u} = 0.$$

$$R\left[\Sigma\alpha_2\frac{\partial\alpha_1}{\partial v}\,.\,\Sigma\alpha_2\frac{\partial\bar{x}}{\partial u} - \Sigma\alpha_2\frac{\partial\alpha_1}{\partial u}\,.\,\Sigma\alpha_2\frac{\partial\bar{x}}{\partial v} + \frac{\partial}{\partial u}\Sigma\alpha_1\frac{\partial\bar{x}}{\partial v} - \frac{\partial}{\partial v}\Sigma\alpha_1\frac{\partial\bar{x}}{\partial u}\right]$$
$$+ \Sigma\alpha_1\frac{\partial\bar{x}}{\partial u}\frac{\partial R}{\partial v} - \Sigma\alpha_1\frac{\partial\bar{x}}{\partial v}\frac{\partial R}{\partial u} = 0.$$

$$R\left[\Sigma\alpha_2\frac{\partial\alpha_1}{\partial v}\,.\,\Sigma\alpha_1\frac{\partial\bar{x}}{\partial u} - \Sigma\alpha_2\frac{\partial\alpha_1}{\partial u}\,.\,\Sigma\alpha_1\frac{\partial\bar{x}}{\partial v} + \frac{\partial}{\partial v}\Sigma\alpha_2\frac{\partial\bar{x}}{\partial u} - \frac{\partial}{\partial u}\Sigma\alpha_2\frac{\partial\bar{x}}{\partial v}\right]$$
$$- \Sigma\alpha_2\frac{\partial\bar{x}}{\partial v}\frac{\partial R}{\partial u} + \Sigma\alpha_2\frac{\partial\bar{x}}{\partial u}\frac{\partial R}{\partial v} = 0.$$

From the definition of $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$, it follows that

$$\alpha_1, \beta_1, \gamma_1 = \frac{E\frac{\partial x}{\partial v} - F\frac{\partial x}{\partial u},\ E\frac{\partial y}{\partial v} - F\frac{\partial y}{\partial u},\ E\frac{\partial z}{\partial v} - F\frac{\partial z}{\partial u}}{\sqrt{E}\sqrt{EG - F^2}},$$
$$\alpha_2, \beta_2, \gamma_2 = X,\ Y,\ Z.$$

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* L. c., p. 323.

By ready calculations we find

$$\Sigma\alpha_2 \frac{\partial \alpha_1}{\partial u} = -\frac{FD}{\sqrt{E}H}, \quad \Sigma\alpha_2 \frac{\partial \alpha_1}{\partial v} = \frac{\sqrt{E}D''}{H}, \quad \Sigma\alpha_1 \frac{\partial \bar{x}}{\partial u} = \frac{tHA}{\sqrt{E}},$$

$$\Sigma\alpha_1 \frac{\partial \bar{x}}{\partial v} = \frac{(1-tb)\,H}{\sqrt{E}}, \quad \Sigma\alpha_2 \frac{\partial \bar{x}}{\partial u} = tD, \quad \Sigma\alpha_2 \frac{\partial \bar{x}}{\partial v} = 0,$$

where we have put, for the sake of brevity,

$$H = \sqrt{(EG - F^2)},$$

$$A = \frac{2E\frac{\partial F}{\partial u} - F\frac{\partial E}{\partial u} - E\frac{\partial E}{\partial v}}{2H^2}.$$

When these values are substituted in the above equations of condition, they become

$$\text{(I)} \quad [R^2 b - (1-tb)\,tE]\,D = 0,$$

$$\text{(II)} \quad \frac{\sqrt{E}DD''t}{H} + \frac{\partial}{\partial u}\left[\frac{(1-tb)\,H}{\sqrt{E}}\right] - \frac{\partial}{\partial v}\left[\frac{AtH}{\sqrt{E}}\right] + \frac{AtH}{\sqrt{E}}\frac{\partial \log R}{\partial v} - \frac{(1-tb)\,H}{\sqrt{E}}\frac{\partial \log R}{\partial u} = 0,$$

$$\text{(III)} \quad \left[\frac{\partial \log t}{\partial v} + \frac{F}{Et} + \frac{\partial \log \sqrt{E}}{\partial v} - \frac{\partial \log R}{\partial v}\right] D = 0.$$

From the forms of (I) and (III), it is evident that they are satisfied when the lines $v = \text{const.}$ are the generatrices of a developable surface. For the present we shall exclude this case, so that the equations (I) and (III) may be replaced by the parentheses equated to zero.

When in (II) the quantity DD'' is replaced by its expression in terms of E, F, G and their derivatives, as given by the Gauss equation, it becomes

$$\text{(II)} \quad tb\frac{\partial \log \sqrt{E}}{\partial u} + b\left(1 + \frac{\partial t}{\partial u}\right) + (1-bt)\frac{\partial \log R}{\partial u} = 0.$$

Solving the equation (I) for R and substituting in (II) and (III), we get

$$\text{(II}'\text{)} \quad \frac{\partial t}{\partial u} = -t\left(2b + \frac{\partial \log E}{\partial u} - \frac{\partial}{\partial u}\log b\right),$$

$$\text{(III}'\text{)} \quad \frac{\partial t}{\partial v} = -\frac{2F}{E} + t\left(\frac{2bF}{E} - \frac{\partial \log b}{\partial v}\right).$$

Differentiating the first with respect to v and the second with respect to u and subtracting, we get

$$\text{(IV)}\qquad t\left(\frac{\partial a}{\partial u}-\frac{\partial b}{\partial v}+\frac{\partial^2 \log b}{\partial u\partial v}\right)+\frac{F}{E}\left(2b+\frac{\partial \log F}{\partial u}-\frac{\partial}{\partial u}\log b\right)=0\,.$$

When the conjugate system on S is composed of the lines of curvature, this equation reduces to

$$\frac{\partial a}{\partial u}-\frac{\partial b}{\partial v}+\frac{\partial^2 \log b}{\partial u\partial v}=0\,. \tag{19}$$

Hence we have the theorem :

When the tangents to the lines of curvature in one system on a surface form a cyclic congruence, it is at the same time a congruence of Ribaucour ; and, conversely.

Recalling some of the results found in the study of congruences of Ribaucour, we have the theorem :

The tangents to the meridians of any surface of revolution form a normal cyclic congruence of Ribaucour.

And

There are no isothermic surfaces for which the tangents to the lines of curvature in both systems form cyclic congruences.

From (IV) we have that, when the parametric lines are not the lines of curvature, the tangents to the curves $v=$ const. form a cyclic congruence of Ribaucour if the functions E, F, G satisfy (19) and

$$2b+\frac{\partial \log F}{\partial u}-\frac{\partial}{\partial u}\log b=0\,. \tag{51}$$

Again, for the tangents to the lines $u=$ const. to form a cyclic congruence of Ribaucour, the functions (E, F, G) must satisfy the equations

$$\frac{\partial a}{\partial u}-\frac{\partial b}{\partial v}-\frac{\partial^2 \log a}{\partial u\partial v}=0\,, \tag{20}$$

$$2a+\frac{\partial \log F}{\partial v}-\frac{\partial}{\partial v}\log a=0\,. \tag{52}$$

If we differentiate (51) with respect to v and (52) with respect to v, and subtract, we get

$$2\,\frac{\partial b}{\partial v}-\frac{\partial^2 \log b}{\partial u\partial v}=2\,\frac{\partial a}{\partial u}-\frac{\partial^2 \log a}{\partial u\partial v}\,,$$

so that, if (51) and (52) are given, and either of (19) and (20), the other follows in consequence of this equation. Hence we may say that *the necessary and sufficient condition that the two congruences of tangents to the curves of a conjugate system on a surface are cyclic congruences of Ribaucour, is either that the curves be orthogonal and equations* (19) *and* (20) *hold, or that equations* (51), (52) *and either* (19) *or* (20) *be satisfied.*

Let S be referred to its lines of curvature; then from (6) we get

$$F_1 = \Sigma \frac{\partial x_1}{\partial u}\frac{\partial x_1}{\partial v} = \frac{E}{b^3}\left(\frac{\partial b}{\partial v} + ab\right)\left(\frac{\partial}{\partial u}\log b - b - \frac{\partial \log \sqrt{E}}{\partial u}\right).$$

Since the function $\left(\frac{\partial b}{\partial v} + ab\right)$ is the invariant k, it cannot be zero unless S_1 is a curve. As this case is excluded, we have that the necessary and sufficient condition that the conjugate system on S_1 be formed of the lines of curvature, and, therefore, that the tangents to the curves $v = \text{const.}$ on S form a congruence of Guichard, is

$$\frac{\partial}{\partial u}\log b - b - \frac{\partial \log \sqrt{E}}{\partial u} = 0.$$

Differentiating with respect to v and making use of (23), we get (19), which leads to the well-known theorem:

The congruences of Guichard are congruences of Ribaucour.

The above equation may be written

$$\frac{\partial}{\partial u}\log b + \frac{\partial}{\partial u}\log\sqrt{G} - \frac{\partial}{\partial u}\log\sqrt{E} = 0,$$

whence we have

$$\frac{\partial \sqrt{G}}{\partial u} = V\sqrt{E},$$

so that

$$\sqrt{G} = V\int\sqrt{E}\,du + V_1. \tag{53}$$

From the above we have that V can have a zero value only in case G is a function of v alone.

In a similar manner we can show that for F_{-1} to be zero we must have

$$\sqrt{E} = U\int \sqrt{G}\, dv + U_1. \tag{54}$$

We have shown in §1 that conditions (53) and (54) cannot be satisfied simultaneously.

When equation (19) and either $F = 0$ or (51) are satisfied, t is given by quadrature from (II′) and (III′), and, consequently, involves an arbitrary constant. Hence, *when a cyclic congruence is also a congruence of Ribaucour, there is an infinity of cyclic systems whose circles have the lines of the congruence for axes.*

In order to consider the case where the cyclic congruence is not a congruence of Ribaucour, we write (IV) in the form

$$\left(\frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u\, \partial v}\right)(1 - 2tb) = \frac{Fb}{E}\left[\frac{\partial \log F}{\partial u} - 3\,\frac{\partial \log b}{\partial u} + \frac{\partial \log E}{\partial u} + 4b\right] - \frac{\partial^2 \log \sqrt{E}}{\partial u \partial v} - \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u\, \partial v} \tag{55}$$

We denote by δ the semi-focal distance and by d the distance from the mean point to the center of the circle. Since it is a characteristic property of cyclic congruences that the focal points lie on the tangents to the lines of curvature of any surface orthogonal to the circle,* we have the relation

$$d^2 + R^2 = \delta^2,$$

hence a real angle σ exists defined by

$$\cos\sigma = \frac{d}{\delta}\,.$$

It is readily found that

$$\cos\sigma = (1 - 2tb),$$

Hence, when the ratio of the right-hand member of equation (55) to the coefficient of $1 - 2tb$ is, in absolute value, less than unity, the congruence of tangents to the curves $v = \text{const.}$ is cyclic. Since the function t corresponding to this case is given by (IV) and consequently doesn't contain any arbitrary quantity, there is only one cyclic system with the lines of the congruence for axes.

The preceding equations are of too complicated a form to enable us to solve the general problem of the derived congruences of a cyclic congruence. We

* Tzitzeica, Thesis.

will therefore close this discussion with an investigation of cyclic systems of equal circles in relation to the above problem.

§7.—*Cyclic systems of equal circles.*

Bianchi has shown that there are only two ways in which cyclic systems of equal circles can be formed; either by describing circles of radius R in the tangent planes to a pseudospherical surface of curvature $-\frac{1}{R^2}$ and with centers at the points of contact, or by drawing tangents to the geodesic lines of curvature of a surface of Monge and with points at the constant distance R from the points of contact as centers describing circles of radius R in the plane of the geodesic, and hence cutting the surface orthogonally.

For the first case the axes of the circles are the normals to the pseudospherical surface. Since these congruences are normal, there cannot be a sequence of derived congruences of this kind.

We pass now to the second method of forming a cyclic system of equal circles. Let the geodesic lines of curvature be $v =$ const. Since these lines are plane, the infinity of circles which meet the surface in points of a line $v =$ const. lie in the same plane and, consequently, their axes have the same direction; from this it follows that the direction-cosines of the lines of the congruence are functions of v alone. Hence, one of the focal sheets will be at infinity and the other will be the envelope of the cylinder, whose right-section is the locus of centers of the circles, when the plane $v =$ const. rolls without sliding upon its generator. As one of the focal sheets is at infinity, there cannot be a suite of derived congruences of this kind. Gathering together the preceding results, we have the theorem:

There cannot exist a sequence of cyclic congruences for which the circles of each congruence are equal.

Incidentally we have been brought to the following result: Given a surface S which is the envelope of a cylinder depending upon a single parameter and of invariable right-section. If in the plane of any right-section, circles of equal radius are described with points of the right section for centers, these circles generate a cyclic system as the cylinder envelopes S and the surfaces cutting the circles orthogonally are surfaces of Monge.

The analytical condition for this is readily found. Thus let S be the envelope of a cylinder and let the congruence of elements of the cylinder have the curves $v =$ const. for edges of regression. The direction-cosines of these lines are

$$\frac{1}{E}\frac{\partial x}{\partial u}, \quad \frac{1}{E}\frac{\partial y}{\partial u}, \quad \frac{1}{E}\frac{\partial z}{\partial u},$$

Since S is the envelope of the cylinder, the lines of the congruence meeting S along the conjugate directions $u =$ const. must have the same directions, so that the derivatives with respect to v of the above direction-cosines must be zero. Thus,

$$\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}\right) = \left(\frac{\partial}{\partial v}\frac{1}{\sqrt{E}} - \frac{a}{\sqrt{E}}\right)\frac{\partial x}{\partial u} - \frac{b}{\sqrt{E}}\frac{\partial x}{\partial v} = 0\,.$$

Since the same equation must be satisfied by y and z, we must have

$$\frac{\partial}{\partial v}\frac{1}{\sqrt{E}} - \frac{a}{\sqrt{E}} = 0, \quad b = 0\,.$$

When a and b are replaced by their expressions (22), it is found that these two equations are the same, namely,

$$E\frac{\partial G}{\partial u} - F\frac{\partial E}{\partial v} = 0\,. \tag{56}$$

Similarly, for the tangents to the curves $u =$ const. to form such a congruence, we must have

$$F\frac{\partial G}{\partial u} - G\frac{\partial E}{\partial v} = 0\,. \tag{57}$$

The necessary and sufficient condition that these two conditions be satisfied simultaneously, is that the point equation of S becomes

$$\frac{\partial^2 \theta}{\partial u \partial v} = 0\,.$$

Hence the theorem:

The tangents to each family of generating curves of a surface of translation form congruences for which one family of the developable surfaces is composed of cylinders.

For the conjugate system on S to be orthogonal and condition (56) be satisfied, G must be a function of v alone; that is, the lines of curvature $u =$ const. must be geodesics, and hence S a surface of Monge. From the properties of these surfaces and a preceding remark, we have the theorem:

Surfaces of Monge are the envelopes of cylinders, of unvariable right-section, depending upon a single parameter. Moreover, if any right-section of this generating cylinder is made and with the points of the section as centers, circles of equal radius are described in the plane of the section; these circles form a cyclic system and the orthogonal surfaces are surfaces of Monge.

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